Brown University
DATA 1010
Practice Midterm I
Instructor: Samuel S. Watson

Name:

You will have three hours to complete this exam. The exam consists of 12 written questions. No calculators or other materials are allowed, except the Julia-Python-R reference sheet.

You are responsible for explaining your answer to **every** *question. Your explanations do not have to be any longer than necessary to convince the reader that your answer is correct.*

I verify that I have read the instructions and will abide by the rules of the exam: ______________________________

Problem 1 **[JULIA]**

Write a Julia function called `appeartwice` which accepts two arguments: a vector `x` and a number `a`. The function should return `true` if `x` has two or more entries which are equal to `a` and `false` otherwise.

```
@assert appeartwice([1,4,1,0,2,1],1) == true
@assert appeartwice([-3,2,-5,7,1],-5) == false
@assert appeartwice([-3,2,-5,7,1],11) == false
```

Make your code as close to correct as you can, but minor syntax errors will be disregarded in the grading.

Solution

Problem 2 **[LINALG]**

Let us say that a vector in a list of vectors is *redundant* if it can be deleted from the list without changing the span of the list. Show that if a list of nonzero vectors is linearly dependent, then the number of redundant vectors is at least two.

Solution

Problem 3 **[MATALG]**

(a) Suppose that A is an $m \times n$ matrix. Explain why a vector $\mathbf{x}$ is orthogonal to the span of the columns of A if it is in the null space of the transpose of A.

(b) Suppose that A is a 10×5 matrix and that $\mathbf{b}$ is a vector which is in the span of the columns of A. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be solved by left-multiplying by A^{-1} to obtain $\mathbf{x} = A^{-1}\mathbf{b}$.

(c) Suppose A is an $n \times n$ matrix and that $\mathbf{b}$ is a vector in $\mathbb{R}^n$. Solve the matrix equation $A\mathbf{x} + \mathbf{b} = \mathbf{x}$ for $\mathbf{x}$ (you may assume matrix invertibility wherever convenient).

Solution

Problem 4 **[EIGEN]**

Recall that eigenvectors corresponding to different eigenvalues are linearly independent (in other words, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$, and if no pair of the λ_i's are equal, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent list). Using this fact, explain why an $n \times n$ matrix has at most finitely many eigenvalues.

Solution

Problem 5 **[OPT]**

Suppose $\mathbf{b} \in \mathbb{R}^n$. For each $\lambda \in \mathbb{R}$, consider the problem of finding the value of $\mathbf{x} \in \mathbb{R}^n$ which minimizes the expression

$$|\mathbf{x} - \mathbf{b}| + \lambda |\mathbf{x}|^2$$

Discuss, qualitatively, the behavior of the solution of this optimization problem as λ ranges over the interval $(0, \infty)$. (Note: do not try to differentiate; approach this one qualitatively from start to finish.)

Solution

Problem 6 **[MATDIFF]**

Suppose that A is an $n \times n$ matrix and that $\lambda \in \mathbb{R}$. Differentiate $A\mathbf{x} + \lambda\mathbf{x}$ with respect to $\mathbf{x}$. Show that the resulting matrix has nonzero determinant for almost all real values of λ (decide on a meaning for "almost all" and state it in your answer).

Solution

Final answer:

Problem 7 **[MACHARITH]**

Which of the following operations results in a number which is exactly equal to `11.0` when evaluated in `Float64` arithmetic?

1. `11.0 + 0.5^30`
2. `11.0 + 0.5^51`
3. `sum([0.125 for i=1:88])`
4. `100.0 + 0.5^48 - 100.0 + 11.0`

Solution

Problem 8 **[NUMERROR]**

Your friend observes that they were able to calculate $A\mathbf{x}$ with error significantly less than $\kappa(A)\epsilon_{\text{mach}}$ (where A is an $m \times n$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^n$). Without knowing further details regarding the A and $\mathbf{x}$ values your friend is using, give two reasons why this might have been the case.

Solution

Problem 9 [PRNG]

Suppose $X_0 \in [0,1]$, and for $n \geq 1$ we define $X_n = \text{mod}(\pi + X_{n-1}, 1)$, where $\text{mod}(x, 1)$ is the difference between x and the greatest integer which is less than or equal to x (so $\text{mod}(5.62, 1) = 0.62$, for example).

(a) Does this sequence have a finite period, and if so, what is the period?

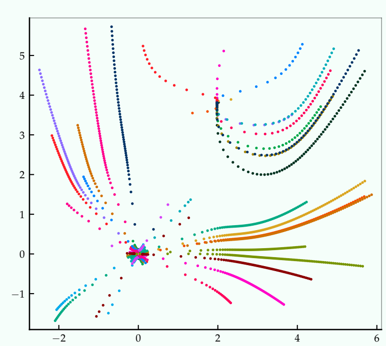
(b) If the values of the sequence are computed iteratively in `Float64` arithmetic rather than real arithmetic, give an upper bound on the period of the resulting sequence.

(c) If we think of the (`Float64`) values $X_0, X_1, X_2, \ldots$ as the output of a pseudorandom number generator, is this PRNG cryptographically secure?

Solution

Problem 10 [NUMOPT]

Several (plain vanilla) gradient descent trajectories are shown for a function $f : \mathbb{R}^2 \to \mathbb{R}$, starting from various points in the square $[-3, 6] \times [-3, 6]$. Describe the graph of f. (For example, how many local minima does it have, and where is the graph steepest?)



Solution

Problem 11 **[PROBSPACE]**

Let $(\Omega, \mathbb{P})$ be a probability space. Use the axioms of probability to show that $\mathbb{P}(A \cap B) \leq \mathbb{P}(B)$ for any events A and B.

Solution

Problem 12 **[CONDPROB]**

Three cards are drawn without replacement from a well-shuffled standard deck. Find the conditional probability that the cards are all diamonds given that they are all red cards. (Note: 13 of the cards are diamonds, 26 of the cards are red, and all of the diamonds are red).

Solution

Final answer: